Private Information in Repeated Auctions*

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Abstract

We study an infinitely repeated two-player game with incomplete information, where the stage game is a first-price auction with pure common values. Before playing, the bidders receive affiliated private signals about the value, which itself does not change over time. Items sold in such an auction environment include bonds, wine, neighboring oil tracts, and wholesale fish. In this setting, learning occurs only through observation of the bids. We show that in the case of one-sided incomplete information, this information is eventually revealed and the seller extracts essentially the entire rent (for large enough discount factors). In contrast, the unique equilibrium with patient players under two-sided incomplete information is purely pooling: no information is ever revealed. In the special case with only two types of each bidder, we are able to fully characterize the equilibrium for all values of the discount factor and all priors.

1 Introduction

This paper analyzes an infinitely-repeated, first-price auction between two players with private information about the common value of all the units. It generalizes Engelbrecht-Wiggans and Weber (1983), which solves the finite-horizon, binary type, one-sided incomplete information case.

We are primarily interested in the optimal strategy of the bidders: how can a bidder exploit his private information without giving it away? How valuable is private information to a bidder? A related concern is the informational content of the prices: are prices revealing and aggregating the players' information? In this sense, our motivation is close to Kyle (1985), who analyses continuous auctions and insider trading, although there is no noise trading in our model.[1]

Information revelation in repeated first-price auctions has already been studied by Hon-Snir, Monderer and Sela (1998), in a model with independent private values. They show that information is eventually fully revealed. However, because "equilibrium analysis of repeated first-price auctions in the framework of repeated games with incomplete information is complex", they assume that players are not rational but use learning schemes.

In contrast with these findings, our results imply that when players are patient and rational, prices do not reveal any information at all. That is, the unique equilibrium (subject to a refinement that actually helps the seller) exhibits the ratchet effect identified in games with one-sided incomplete information by Freixas-Guesnerie-Tirole (1985) and Hart-Tirole (1988). Here, both bidders make low bids consistent with their worst private

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[1] Another difference is that in his model, bidders choose quantities and the price clears the markets. Because in his model bidders foresee how their action affects the price, just as in our model bidders foresee how their action affects their probability of winning, this distinction is merely interpretative.

information, even when their estimate is high, and prefer to win half of the time at such a low price, rather than break the tie in their favor and divulge thereby some of their information.

This result may appear surprising to the reader familiar with Engelbrecht-Wiggans and Weber (1983). After all, these authors establish that, with one-sided incomplete information, information is eventually fully revealed, and we show that their insight carries over to the infinite-horizon case and to a more general set-up. However, we also show that there is a significant difference between one-sided incomplete information and two-sided incomplete information, even if beliefs are degenerate.

Our results imply that, for repeated first-price auctions, such as treasury bills, the auctioneer's revenue from the sequential procedure is lower than under a one-time batch sale if bidders are patient enough, or alternatively, the frequency of auctions is very high.

This paper builds on Engelbrecht-Wiggans and Weber (1983), which investigates the undiscounted, finite-horizon version of this game in which only one player has binary, private information. One of their striking conclusions is that being uninformed is an advantage if the horizon is long enough. Also, the seller's expected revenue is larger under the one-stage batch sale than under the sequential auction if the initial probability of a high valuation is sufficiently large. Our results imply that their main conclusions do not rely on backward induction and remain valid when the horizon is infinite and players are sufficiently patient, because, as will be seen, when incomplete information is one-sided, a date T is endogenously determined such that, along any equilibrium outcome, a high-signalled bidder has revealed his information by time T.

Two other papers are closely related to his one. First, Hausch (1986), building on Ortega-Reichert (1968), considers a two-person, two stage first-price auction with common values. At the beginning of the game, bidders receive conditionally independent signals about the identical value of the two units. The value of the objects to the players is additive. Thus, Hausch's setting corresponds to the two-period, undiscounted symmetric version of this model (but it is richer inasmuch as signals are not necessarily perfectly informative). He shows that it is impossible that a high-signalled player's bid always reveals his signal in the first auction. Instead, either he always conceal his signal in the first auction by bidding like a low-signalled player (pooling), or he randomizes between concealing and revealing his signal in the first auction (semi-pooling). When each object is worth one with probability p, and 0 otherwise, the second possibility arises for p sufficiently large.

Second, Bikhchandani (1988) examines a finitely repeated *second-price* auction with common values between two bidders. The value of the objects to the players is additive, and identically and independently drawn from a common distribution. Both players receive conditionally independent signals about the value of each unit, but one of the bidders is of one of two possible types (and this is private information). The strong type's valuation of any given unit is higher, by a multiplicative constant, than the low type's valuation. Because winning is especially bad news for the uninformed player when his opponent may be of the high type, the winner's curse is intensified for the uninformed player, forcing him to submit lower bids in equilibrium. This weakens the winner's curse for the informed bidder of the ordinary type and decreases the price he has to pay whenever he wins. The ordinary type has therefore an initial incentive to mimic the strong type by making high bids.

Finally, a more general analysis of repeated games with incomplete information has been carried out by Aumann, Maschler and Stearns (1966-68).

It may be worthwhile to point out that several goods are sold in repeated first-price auctions, such as various types of government bonds and red Bordeaux wines. See Ashenfelter (1989) for more examples. Pezanis-Christou (2000) has collected and analyzed evidence from a fish market with asymmetric buyers (retailers vs. wholesalers). As he points out, fish markets are characterized by supply uncertainty.

Section 2 presents the basic model. Section 3 characterizes the equilibrium for the general case of one-sided incomplete information. Section 4 proves our central result that the unique equilibrium (for large δ) in the two-sided case is purely pooling. Section 5 explores the special case with only two types of each bidder, where we are able to describe the outcomes for general δ, and prove uniqueness in the symmetric case for all δ and all priors p. Finally, section 6 concludes briefly.

2 The Model

This paper considers an infinitely-repeated game between two risk-neutral bidders, player 1 and player 2. In every period $t = 0, 1, \ldots$ one indivisible unit is sold using a sealed-bid first-price auction. In case of a tie, each player wins with equal probability. We assume common values:

Assumption 1: Both bidders value each unit identically.

All units auctioned off have the same value, represented by the random variable V. That is, we assume perfect correlation over time:

Assumption 2: The value of the units from one period to the next is perfectly correlated.

Before the game starts, each bidder i ($i = 1$ or 2) receives a signal S_i concerning the value of the object. The signal can take $m+1$ different values for player 1, and $n+1$ different values for player 2, that is, $S_1 \in M = \{0, 1, \ldots, m\}$, $S_2 \in N = \{0, 1, \ldots, n\}$. Signals are also referred to as types. A high (low) signal is statistical evidence for a high (low) value of the object. This is formalized by:

Assumption 3: The variables V, S_1 and S_2 are affiliated.

Affiliation is developed in Milgrom and Weber. Let $x = (v, j, k)$, $x' = (v', l, m)$ be elements of $\mathbb{R} \times M \times N$. Let $\bar{x}$ and $\underline{x}$ be respectively the componentwise maximum and componentwise minimum of x and x'. Then if (V, S_1, S_2) has the joint density f, these variables are affiliated if, for all x and x', $f(\bar{x}) f(\underline{x}) \geqq f(x) f(x')$. This implies that $v(j,k) \triangleq E[V \mid S_1 = j, S_2 = k]$ is weakly increasing in each of its arguments. Let $p(j,k)$ denote the probability that $S_1 = j$ and $S_2 = k$. Finally,we denote by $E_k[v(j,k)]$ the expected value of a type j player 1:

$$E_k[v(j,k)] = \frac{\sum_{k=0}^{n} p(j,k) v(j,k)}{\sum_{k=0}^{n} p(j,k)}$$

and similarly for $E_j[v(j,k)]$. The last assumption on parameters can be dispensed with at the expense of trivial complications.

Assumption 4: (i) The function $v(j,k)$ is strictly increasing in each of its arguments; (ii) $v(0,0) \geqq 0$; (iii) $p(j,k) > 0$ for all $(j,k) \in M \times N$.

Players maximize their payoff, which is the discounted sum of their profit in each auction, using a common discount factor $\delta \in [0,1)$. Therefore, their utility does not exhibit "diminishing marginal returns" for winning more units. We emphasize that players do not learn the value of a unit upon buying it. Learning, therefore, is restricted to inferring one's rival's information.

The solution concept used is Perfect Bayesian Equilibrium (P.B.E.). We further wish to prune any equilibrium which depends on payoff irrelevant information, and therefore assume that continuation play only depends on (all players') current beliefs. That is, if there exist two histories after which every type of every player entertains the same beliefs about its opponent, then the actions that follow these histories are the same as well.[2] In addition, we impose two refinements.

First, we assume "no underbidding". Under perfect information, if all units are known to be worth $v > 0$, there exist many equilibria. For instance, for any $b \in [0, v]$, there exists an equilibrium in which both players bid b repeatedly on the equilibrium path. Because our focus is not on collusion *per se*, we assume that bids are at least as large as what the unit is commonly known to be worth, conditional on winning. In this example, "no underbidding" requires bids to be at least v in every period. Note that this refinement benefits the seller, and yet one of our main findings is that the auctioneer's revenue is low.

Second, we assume "no overbidding". That is, a bidder observing a bid b or b_+ (see below for explanation of + bids) assigns henceforth, if possible, zero probability to all types of his opponent for which such a bid strictly exceeds the maximal possible value of a unit, given their type and beliefs.

A bid is a *winning* bid if it is equal to the highest bid with strictly positive probability. Formally:

Assumption 5: In period t,

[2]If this assumption is not imposed, additional "reputational" equilibria exist, as discussed in the concluding comments.

(i) (stationarity) Actions chosen only depend on current beliefs.[3]

(ii) (no underbidding) The lowest winning bid is at least equal, for some player's type assigned positive probability, to his expected value of the unit conditional on winning with this bid. Also, if b is an equilibrium bid, then there is a winning bid no larger than $b+\varepsilon$, for any $\varepsilon>0$.

(iii) (no overbidding) Upon observing a bid b, i's beliefs assign thereafter zero probability to any type for whom b is strictly more than the (conditional) essential supremum of V.

Assumptions (i) and (ii) are necessary for our uniqueness result, and other equilibria that can be supported in their absence are described in the concluding comments. We do not know whether more equilibrium outcomes are introduced if we drop assumption (iii).

Finally, given our tie-breaking rule, we cannot always guarantee existence under the assumptions above. In that case, we allow for a bid b_+ that is infinitesimally larger than b. One can think of b_+ as b "plus a penny". Although a player's utility from a winning bid of b is the same as from a winning bid of b_+, a bid of b_+ is strictly larger than a bid of b, so that the distinction matters in light of Assumption 5(iii). Such a nonstandard bid b_+ is only necessary for b the lowest bid made with positive probability, so that, unless specified otherwise, all symbols and commonly used concepts (support, interval, etc.) refer to the real line and its standard structure. Engelbrecht-Wiggans and Weber (1983) introduce the same idea, and it is indeed exactly in our extension of their model that we will require it. Any claims of uniqueness that we make for equilibria that utilize nonstandard bids are, of course, within the class of all strategies using such bids anywhere in the support.

3 One-sided Incomplete Information

Suppose that M has $m+1>1$ elements, while N has only one element. Accordingly, we refer to player 1 as the informed player, or player I, and to player 2 as the uninformed player, or player U, and we write $v(j)$ instead of $v(j,0)$, where $(j,0)\in M\times N$. Given Assumption 5(ii), normalize $v(0)$ to 0. The solution to the one-sided static model is a special case of the two-sided version (see section 4.3).

Given an equilibrium, let H_t^* be the subset of $t-$histories H_t (including the null history) that have positive probability under the equilibrium strategies, and let $H_t'\subseteq H_t^*$ be the set of histories in which the bids of the informed bidder have been equal to zero for all periods up to t. Let γ_i^t be the probability with which player I is of type i and makes a strictly positive bid in period t, conditional on some $h_t\in H_t'$, and let p_i^t be the probability that player I is of type i, conditional on the same event. Because of 5(i), γ_i^t does not depend on the specific $h_t\in H_t'$. By Bayes' rule,

$$p_i^{t+1}=\frac{p_i^t-\gamma_i^t}{1-\sum_{i=0}^m\gamma_i^t},$$

where $p_i^0=p_i$. By 5(iii), the informed bidder's type 0 bids 0 with probability one, so that $\gamma_0^t=0$, all t, and accordingly $1-\sum_{i=0}^m\gamma_i^t=p_0^t/p_0^{t+1}$. For $i>0$, let $T_i=\inf\left\{t\in N;p_i^{t+1}=0,\ \forall h_t\in H_t^*\right\}$, that is, T_i is the length of the longest equilibrium history in which, given that the informed bidder is of type i, all bids by the informed bidder have been 0. Let π_i^t be the (normalized) payoff of the informed player's type i from period t on, given $h_t\in H_t^*$. Let U_t denote the uninformed bidding distribution in period t conditional on the same event and π_U^t the corresponding (normalized) payoff. Finally, let S_i^t be the support of the bid distribution by the informed player's i type, in period t, conditional on the same event. Let $S^t=\cup S_i^t$, let S_U^t be the uninformed player's bid distribution support in period t given $h_t\in H_t'$ and set $\beta_t=\max\{x\in\mathbb{R}\mid x\in S_U^t\}$. We prove that:

Theorem 1 *$\exists\delta'<1$: $\forall\delta\in(\delta',1)$, $T_i=\max_j T_j\triangleq T<\infty$, for all $i\in M\backslash\{0\}$. In addition, $\lim_{\delta\to1}T=\infty$, $\lim_{\delta\to1}\delta^T=1$, $\lim_{\delta\to1}\pi_U^0=0$, and $\lim_{\delta\to1}\pi_U^0/\pi_i^0=\infty$, all i.*

[3] Formally, we mean beliefs of all orders (so that the assumption is as weak as possible), but note that in any case we will use this only when beliefs on one side are [commonly known to be] degenerate.

This theorem states that, as players become more patient, type $i > 0$ of the informed player can bid 0 in up to but no more than T periods, where T is independent of his type. Although $\lim_{\delta \to 1} T = \infty$, $\delta^T \to 1$, so that players' payoffs depend essentially on their payoff after T. Both players' normalized payoff tends to zero, but the uninformed bidder still fares better than the informed bidder.

Proof. This result will be proved in several steps.

1. $\pi_m^0 > 0$: by 5(iii), an informed bidder with signal i does not bid more than $v(i)$. If $\beta_0 \leqq v(m-1)$, then $\pi_m^0 \geqq (1-\delta)(v(m) - v(m-1)) > 0$. If $\beta_0 > v(m-1)$, then β_0 outbids all types of the informed player but the highest, so that, conditional on winning, the unit is worth at most the (unconditional) expected value of V, EV, and because the uninformed player could bid 0 instead, $\beta_0 \leqq EV$, so that $\pi_m^0 \geqq (1-\delta)(v(m) - EV) > 0$ as well.

2. $\forall h_t \in H_t'$ such that $t < T$, $U_t(0) = 0$: indeed, bidding 0_+ yields a higher payoff than 0, since, conditional on the informed bidder's bid being 0, the unit's expected value is strictly positive, and the uninformed bidder's continuation payoff does not depend on his own bid, by 5(i).

3. $T_i < \infty$, all $i \in M$: suppose that $T_m = \infty$. Then the informed bidder's type m is willing to bid 0 in every period, which yields a payoff of 0, as $U_t(0) = 0$ for all t. This contradicts $\pi_m^0 > 0$, by (1). In turn, this implies that $\pi_{m-1}^0 \geqq \delta^{T_m}(1-\delta)\left(v(m-1) - \max\left(\sum_{i<m} p_i^t v(i), v(m-2)\right)\right) > 0$, so that, by the same argument, $T_{m-1} < \infty$. An easy induction establishes now the result.

4. $\forall h_t \in H_t', \beta_t > 0_+$: otherwise, the lowest, strictly positive type of the informed bidder which still has positive probability given h_t must bid slightly more than β_t with probability one, so that the uninformed bidder could profitably deviate by bidding more.

   Let $\alpha_t = \max\{x \in \mathbb{R} \mid x \in S^t\}$. $\forall h_t \in H_t', \alpha_t > 0_+$, since $\beta_t > 0_+$ and the uninformed bidder would gain otherwise by bidding slightly less than β_t. Define also $\tau = \inf\{t \in N; \beta_t \in S_1^t\}$.

5. If $b \in S_1^t$, $b > 0$, then $(0,b) \subset S_1^t$: otherwise, there exists either some interval $I \subseteq (0,b)$, $I \cap S_t = \emptyset$, or there exists $b' \in (0,b)$ such that $j = \min\{k \in M \mid b' \in S_k^t\} > 1$. The former is impossible by the standard argument (at least one of the players bidding just slightly above I would gain by bidding less), and the second would violate the payoff's single-crossing property, given that the informed bidder's type 1 (resp. type j) has no continuation payoff from bidding b (resp. b').

6. $\forall 1 \leqq t \leqq T$, $U_t(0_+) > U_{t-1}(0_+)$: otherwise, if $U_{t-1}(0_+) \geqq U_t(0_+)$, there exists $\varepsilon > 0$ sufficiently small so that the informed player's lowest type bidding ε in period t would gain from bidding ε in period $t-1$ instead. This implies that, in particular, for $\varepsilon > 0$ sufficiently small, $\varepsilon \in S_1^{t-1}$ implies that $\varepsilon \in S_1^t$: otherwise one derives a contradiction as in (5) by considering $\varepsilon \in S_1^{t-1}$ and $\varepsilon \in S_j^t$, for $j = \min\{k \in N \mid b \in S_k^t\} > 1$.

7. $T_1 = T$: if $b \in S_1^t$, $b > 0$, and $t < T$, then $(0,b) \subset S_1^t$ by (5), and (6) implies that, for $\varepsilon > 0$ small, $\varepsilon \in S_1^{t+1}$.

8. $\tau < \infty$. If $\max_{i>1} T_i < T$, then this is obvious. If not, it must also be that $\beta_T \in S_1^T$: Suppose not, that is, suppose that $\gamma = \max\left\{x \in \mathbb{R} \mid x \in S_1^T\right\} < \beta$. By definition of T, $\gamma > 0$. Recall that by (5), this implies that $(0,\gamma) \subset S_1^T$, and therefore, for every $b < \gamma$, the uninformed player's posterior upon observing such a bid assigns positive probability to type 1. For all $\varepsilon > 0$ sufficiently small, there exists a lowest type j (possibly depending on ε) of the informed player for which $\gamma + \varepsilon$ is in his bidding support. We claim that there exists ε small enough so that this type j would benefit from bidding $\gamma - \varepsilon$ instead of $\gamma + \varepsilon$. Because U_T is continuous on $(\gamma - 2\varepsilon, \gamma + 2\varepsilon)$, it is sufficient to show that the continuation payoff from bidding $\gamma - \varepsilon$ is bounded away from zero (because j is the lowest type bidding $\gamma + \varepsilon$, his continuation payoff from such a bid is zero). This is trivial if type 1 is the only type bidding $\gamma - \varepsilon$, and follows from straightforward arguments in general (the bid support of the uninformed player in such a subgame has $v(1)$ has a lower extremity, and

its distribution function, evaluated, say, at $[v(1)+v(j)]/2$ is bounded away from zero, ensuring that such a bid yields strictly positive profit to type j).

9. $\lim_{\delta\to 1}\delta^{T-\tau}=1$. Clearly, $\forall\varepsilon>0$, $|\alpha_t-\beta_t|<\varepsilon$ for all t, $h_t\in H_t'$ (the lowest type of the player bidding the higher of the two would gain by bidding less). In addition, neither player's bidding distribution can have an atom at the maximum of its support: if the uninformed bidder had such an atom, the informed bidder's lowest type j for which $\alpha_t\in S_j^t$ would gain by bidding $\alpha_t+\varepsilon$ instead; if the informed bidder had, then because all informed bidder's types j making such a bid must have valuation $v(j)>\beta_t$, the uninformed bidder would gain by bidding $\beta_t+\varepsilon$ instead, for $\varepsilon>0$ small enough. Therefore, if the uninformed bidder bids β_t, he wins with probability one. For any t such that $\tau\leqq t\leqq T$, and any $h_t\in H_t'$, β_t must be decreasing in t, because if $\beta_t<\beta_{t+1}$, the lowest type of the informed bidder 's type i whose support S_i^{t+1} includes β_{t+1} would strictly gain from bidding β_t in period t instead. Because the uninformed bidder is indifferent between bidding β_t and 0_+ in any such period,
$$\left(1-\sum_{i=1}^{m}\gamma_i^t\right)\sum_{i=1}^{m}p_i^{t+1}v(i)=\sum_{i=1}^{m}p_i^t v(i)-\beta_t,$$
which implies that $\beta_t=\sum_{i=1}^{m}\gamma_i^t v(i)=\sum_{i=1}^{m}\left(p_i^t-p_0^t p_i^{t+1}/p_0^{t+1}\right)v(i)$. Because p_i^t/p_0^t is decreasing in t for all i, $\beta_t>\varepsilon>0$ implies
$$v(m)\sum_{i=1}^{m}\left(\frac{p_i^t}{p_0^t}-\frac{p_i^{t+1}}{p_0^{t+1}}\right)\geqq\sum_{i=1}^{m}\left(\frac{p_i^t}{p_0^t}-\frac{p_i^{t+1}}{p_0^{t+1}}\right)v(i)>\varepsilon/p_0^t>\varepsilon,$$
so that, if $\beta_t>\varepsilon$ all $t\in\left[\tau,\tau+\hat{T}\right]$, then $\kappa\geqq\sum_{i>0}p_i^{\tau+\hat{T}}/p_0^{\tau+\hat{T}}+\hat{T}\varepsilon/v(m)$, so that $\hat{T}\leqq v(m)\kappa/\varepsilon$. Hence, for all $\varepsilon>0$, $\pi_1^\tau\geqq(1-\delta)\delta^{v(m)\kappa/\varepsilon}(v(1)-\varepsilon)$, which tends to $(1-\delta)(v(1)-\varepsilon)$, as δ tends to 1. Because this argument is independent of ε, $\pi_1^\tau/(1-\delta)\to v(1)$ as $\delta\to 1$, and because the informed bidder's type 1 is willing to bid 0 instead for $T-\tau$ additional periods, it must be that $\lim_{\delta\to 1}\delta^{T-\tau}=1$.

10. $\tau=0$: otherwise, the lowest type bidding $\beta_{\tau-1}$ after a history $h_{\tau-1}\in H_{\tau-1}'$ would strictly gain from bidding 0 in the following $T-\tau+1$ periods, given the arguments developed in (8).

11. $\lim_{\delta\to 1}\delta^T=1$ follows from (8) and (10).

12. $\lim_{\delta\to 1}T=\infty$: because $\lim_{\delta\to 1}\pi_i^0/(1-\delta)\geqq v(i)$ for all $i\in M$ by (11), it must be that $\lim_{\delta\to 1}\alpha_t=0$ $\forall h_t\in H_t'$, and therefore also $\lim_{\delta\to 1}\beta_t=0$, implying that $\lim_{\delta\to 1}\gamma_i^t=0$, all $i\in M$, and therefore $\lim_{\delta\to 1}p_i^{t+1}=p_i^t$. If $\lim_{\delta\to 1}T<\infty$, then $\lim_{\delta\to 1}p_i^T=p_i^0$, $\forall i\in M$. Hence, by definition of T, $\lim_{\delta\to 1}\gamma_i^T=p_i^0$, and thus $\lim_{\delta\to 1}\beta_T=\lim_{\delta\to 1}\sum_{i=1}^{m}\gamma_i^t v(i)=\sum_{i=1}^{m}p_i^0 v(i)$ which does not tend to zero, a contradiction.

The other conclusions follow immediately from (11) and (12). ■

4 Two-sided Incomplete Information

Suppose now that both M and N have at least two elements. In this section, we establish that there exists a unique equilibrium outcome if players are sufficiently patient. On the equilibrium path, both players fully pool: they bid λ independently of their type, where λ is defined as
$$\lambda=\min\langle E_k[v(0,k)],E_j[v(j,0)]\rangle.$$
Thus λ is the smallest expected value that any type of any player has for the object.

4.1 Existence

Theorem 2 $\exists \delta' < 1$*:* $\forall \delta \in (\delta', 1)$*, it is a Perfect Bayesian Equilibrium (PBE) outcome (satisfying our assumptions) for all types of both players to bid* λ *forever along the equilibrium path.*

Proof. If beliefs are one-sided degenerate on either type m or type n, the equilibrium play is as in the one-sided model described in the previous section (not pooling). If beliefs are one-sided degenerate on some lower type k of player 2 (say), then there is instead a pooling equilibrium at the bid $\lambda = v(0, k)$. This shows clearly the distinction between knowledge and belief! To see that such an equilibrium exists, simply assume that any player who bids more than λ is thought to be the highest type possible (of that player), so that bids immediately jump. For large enough δ, this leads to payoffs that are essentially zero (at most), whereas all types of both players (except player 1 type 0) are making strictly positive profits along the equilibrium path (that are bounded away from zero independently of δ). For type 1^0, who expects zero profits anyway, bidding above λ leads to a loss in the current period (because his expected value for the good is precisely λ) and continuation payoffs of exactly zero (since all bids from then on will be above his value), so there is no incentive for him to deviate either.

In the general two-sided case with non-degenerate beliefs, we can use similar strategies to support the equilibrium. Assume that any player bidding above λ is believed thereafter to be of the highest type (i.e. either m or n respectively), leading to the one-sided equilibrium of the last section and hence continuation payoffs that approach zero (for both players), at most, as δ goes to 1. Since all but possibly the lowest types of each player are making profits that are bounded away from zero (independently and hence uniformly in δ) in equilibrium, they will not deviate. But exactly analogously to above, any low type making zero profits can only lose by overbidding in the current period, and will face continuation payoffs of zero anyway, so has no reason to deviate either.

To see that these updating protocols satisfy our refinements, first note that λ is exactly the minimum bid required by 5(ii). The extreme updating (degenerate on the highest type) is stricter than that required by 5(iii), but goes in the same direction and certainly does not violate it. This particular updating is not unreasonable (the highest type has the strongest incentive to try to win at any stage), but note that the equilibrium outcome survives with much weaker protocols: any revised distribution that first-order stochastically dominates the prior (i.e. such that high bids are "good news" about your opponent's type) will lead to larger expected values and thus a higher pooling bid in the continuation. For high enough δ, this outweighs any possible one-shot gains. Finally, stationarity is clear. ■

4.2 Uniqueness

There are important differences between the true one-sided asymmetric information case considered above, and a two-sided asymmetric environment with degenerate beliefs on one player, which is what we now turn to. Let us refer to the support of the equilibrium bid distribution of player i's type k as player i's type k bidding support, and to the union of these over all types k (in the support of his opponent's beliefs) as player i's bidding support. We introduce the additional requirement that for each player $i = 1, 2$, player i's bidding support should be connected. Observe that this does not require that player i's type k bidding support be connected, nor does it impose any restriction whatsoever on the relationship between bids and types. This assumption rules out some equilibria that we find unconvincing, an example of which is briefly described in the concluding comments.

Assumption 5 (iv) (connected supports) Each player i's bidding support is connected.

The intuition behind the proofs of the following results is fairly straightforward. For the case of degenerate beliefs on one player, if the informed player I is not pooling then he must be making a range of bids. In that case the uninformed player U is doing the same, so I enjoys no flow payoff from pooling. Hence there is a last period in which he is willing to pool, but that implies that at least one type of player I would prefer to cheat and mimic the low type from then on rather than separate and be the new low type (with zero payoff). In the general two-sided case, the basic idea is that separating will lead to either more optimistic beliefs by the other player and thus a higher pooling bid level in the future (which offsets any one-shot gains for large δ) or to the one-sided equilibrium of section 3 in which both players make arbitrarily low payoff for large δ. Even if your opponent is planning to separate (which is definitely bad for you, but of course you have no way to affect it), your best option is to pool and keep the new pooling bid relatively low (or in the worst case receive Π^I, which is still your best outcome if he does fully reveal as a high type).

Theorem 3 $\exists\delta' < 1$*:* $\forall\delta \in (\delta', 1)$*, if beliefs are one-sided degenerate (not on type m or n), then the pooling outcome of Theorem 2 is the unique outcome among all PBE satisfying our assumptions.*

Proof. Recall that if beliefs are degenerate on one of the highest types (m or n), then we play as in the true [with certainty] one-sided equilibrium of section 3, in particular not pooling. Here, for concreteness, we take beliefs to be one-sided degenerate on some type $k < n$ of player 2. In this case, the minimum expected value λ (as defined previously) is $v(0,k)$. We proceed in several steps.

1. Player 1 type 0 bids only λ after any history and makes no profit: by 5(ii) no type of any player will ever bid less than λ, so the type 0 can never hope for positive profits. By bidding above λ, he risks winning (because $\lambda + \varepsilon$ is a winning bid by 5(ii)) and thereby making a loss.

2. If all types of player 1 use the same bid distribution strategy (not necessarily stationary), then it is degenerate on λ at all times (this follows immediately from **1**) and player 2 does the same thing: first we note that by stationarity player 2 will have the same strategy in every period. Player 2 cannot bid strictly above λ with probability one or player 1 would never win and would thus make zero profits (whatever his type), which is impossible. If player 2 bids both λ and $b > \lambda$ with positive probability, then by connectedness (i.e. 5(iv)) he also bids $\lambda + \varepsilon$ with positive probability, for any $\varepsilon > 0$ small enough. However, he would then strictly prefer such a bid $\lambda + \varepsilon$ to the bid λ (since player 1 puts a mass at λ and player 2's continuation value is the same for all equilibrium bids), a contradiction. So player 2 bids λ with probability one instead, as claimed.

3. If there is ever any period t in which all types of player 1 (in the support of player 2's beliefs) bid only λ, then both players do so from then on: this follows directly from **2** and stationarity, noting that beliefs do not change after such a period t.

4. Let β_t be the maximum over the bidding supports of players 1 and 2 in period t, assuming that player 1 has bid only λ so far (if not, we have a new lowest type for player 1 and a new lowest bid λ'). Then $\beta_t > \lambda$ implies that player 2 puts no weight on λ (in period t): certainly player 2 is not bidding only λ, otherwise the lowest type of player 1 who bids above λ (and who then receives zero continuation payoff) would want to bid an arbitrarily small amount above λ. In fact, this implies that 2's connected support must have λ as its infimum. But given that player 2 does bid above λ in equilibrium, the same reasoning as above shows that player 2 would strictly prefer to bid just above λ than at λ.

5. We cannot have $\beta_t > \lambda$ for an infinite number of periods: if so, by **3**, it would have to hold in every period, but that would mean that at least some type $j > 0$ of player 1 was willing to bid λ in every period and always lose (by **4**). This would yield zero profits, a contradiction.

6. There can be no last period T in which $\beta_T > \lambda$: by the same reasoning as in **2**, player 2 must bid λ from period $T + 1$ on (note that this argument depended on the existence of types $j > 0$ of player 1, but without them 5(iii) still implies that player 2 won't bid above λ). But now consider the smallest type of player 1 who bids above λ with positive probability in period T. Such a bid leads to zero continuation payoff for this type, so for large δ he would strictly prefer instead to pool at λ from period T on.

7. Putting **5** and **6** together, we conclude that there can be no period in which $\beta_t > \lambda$. So player 1 always bids λ and, by **2**, we're done. ■

To obtain uniqueness in the two-sided case when beliefs are nondegenerate, the affiliation property must be strengthened. More precisely, we assume henceforth that types are conditionally independent, as is often assumed in the literature with common-values.

Assumption 3': The variables S_1 and S_2 are pairwise affiliated with V, but, conditional on a realization of V, they are independent.

Lemma $\exists\delta' < 1$: $\forall\delta \in (\delta', 1)$, *if beliefs are two-sided nondegenerate and the support of beliefs never changes along the equilibrium path, then under Assumptions 1-2,3',4-5, any equilibrium must involve full pooling (i.e. both players' bidding supports are degenerate).*

Proof. Let $\mu_j(k)$ be the probability assigned by player 1's type j to his opponent being of type k. Because of independence, $\mu_j(k)$ is independent of j. So suppose that both type j and type $j' > j$ are indifferent between all equilibrium bids, as they must be if the support of player 2's beliefs does not change. Consider the two strategies consisting in always submitting the highest such bid, and always submitting the lowest such bid. Let the total discounted expected payment from such strategies be respectively $\bar{b}$ and $\underline{b}$.[4] Let $p_k(b)$ be the total expected discounted probability that player 1 wins against player 2's type k when 1 employs the strategy that corresponds to the payment $b = \bar{b}$, $\underline{b}$. Note that this is the same number for j and j' since they agree on player 2's strategy and distribution of types (by independence). Then we must have, for $i = j, j'$,

$$\begin{aligned}\sum_k \mu(k)\, p_k\left(\bar{b}\right)\left(v(i,k) - \bar{b}\right) &= \sum_k \mu(k)\, p_k(\underline{b})\left(v(i,k) - \underline{b}\right), \text{ or}\\ \sum_k \mu(k)\left(p_k\left(\bar{b}\right) - p_k(\underline{b})\right) v(i,k) &= \sum_k \mu(k)\left(p_k\left(\bar{b}\right)\bar{b} - p_k(\underline{b})\,\underline{b}\right).\end{aligned}$$

Since the RHS is the same for j and j', we get

$$\sum_k \mu(k)\left(p_k\left(\bar{b}\right) - p_k(\underline{b})\right)\left(v(j',k) - v(j,k)\right) = 0.$$

Now note that if $\bar{b} \neq \underline{b}$, then $p_k\left(\bar{b}\right) > p_k(\underline{b})$ for all k because $\bar{b}$ corresponds to strictly higher bids and all types of player 2 are making all equilibrium bids with positive probability (by Theorem 4). But $v(i,k)$ is strictly increasing in i by Assumption 4, so we have a contradiction unless $\bar{b} = \underline{b}$, which implies that player 1 is fully pooling on one bid. The same argument works for player 2, and then it is immediate that these two bids are the same (and they both must equal λ by assumption 5(ii) as before). ■

Theorem 4 $\exists\delta' < 1$: $\forall\delta \in (\delta', 1)$, *if beliefs are two-sided nondegenerate, then the only PBE outcome satisfying our assumptions is the pooling equilibrium described in Theorem* 2.

Proof. By the lemma and the obvious fact that players can't fully pool at any bid other than λ, it suffices to show that the support of beliefs never changes, i.e. that every equilibrium bid by player i is made by all types of that player. We proceed by induction on $m+n$. In particular, any bid that is not made by all types immediately leads to the relevant full pooling equilibrium via the inductive hypothesis. By Theorems 1 and 3, we may assume the first step of the induction.

[Proof under revision; most recent version available from the authors upon request.] ■

4.3 Static Benchmark

Because we have established that the equilibrium outcome is pooling when the discount factor is close to one, and the expected revenue is therefore simply $v(0,0)$, it may be useful to give the corresponding expected revenue for the case of a one-stage batch sale, that is, to solve for the static expected revenue. Consider the case in which $\delta = 0$, that is, bidders play the static, first-price sealed bid auction in every period. It is not known

[4] Note that these strategies do not necessarily yield the maximum and minimum expected discounted probabilities of winning against player 2's strategy.

whether an equilibrium of such an auction exists under Assumptions 1, 3 and 4. One has to assume further either that: (i) $M = N$ and $p(i \mid j) = p(j \mid i)\ \forall i, j \in M$ (see Hausch (1987) for details); (ii) types are (conditionally) independently distributed; or (iii) M and N have at most two elements.

Under assumption (i), Hausch has derived explicit formulae for the ex ante profits to each player. Strategies are symmetric, although this assumption does not imply that the model is symmetric. Under assumption (ii), Wang (1991) has characterized the equilibrium, provided the model is symmetric. Maskin and Riley (2000, Proposition 2) prove existence of a monotonic equilibrium when ties are broken using a Vickrey auction. An equilibrium is monotonic, if, for any two types $i > i'$ of any player, $b_i \in S_i$, $b_{i'} \in S_{i'}$ implies $b_i \geqq b_{i'}$, where S_i, $S_{i'}$ are the supports of the bidding distributions of, respectively, type i and type i'. We now give a complete characterization of this (unique) monotonic equilibrium under independence. Without loss of generality, we normalize $v(0,0)$ to 0.

Because there are only two players, the supports of the bidding distributions of two different types of the same player intersect in one point if these types are consecutive. For each player and every pair of consecutive types of this player, let $\alpha(k)$ be this bid, where arguments are picked in any way such that $\alpha(1) \leqq \cdots \leqq \alpha(m+n)$. Let S_i^1 denote the support of the bid distribution of player 1's type i, and analogously S_j^2 for player 2's type j. If k is the index that corresponds to the intersection of the supports of player 1's (resp. player 2's) type i and $i+1$ (resp. type j and $j+1$), let $m(k) = i+1$ and $n(k) = \max\left\{j \in N \mid \alpha(k) \in S_j^2\right\}$(resp. $n(k) = j+1$ and $m(k) = \max\left\{i \in N \mid \alpha(k) \in S_i^1\right\}$). Let $\alpha(0) = 0$, $m(0) = n(0) = 0$, and denote the highest bid in either player's support by $\alpha(m+n+1)$. Let $p(i)$ (resp. $q(j)$) be the probability that player 1 (resp. player 2) is of type i (resp. of type j), and let $F_{m(k)}$ (resp. $G_{n(k)}$) be the bid distribution of player 1's type $m(k)$ (resp. player 2's type $n(k)$) on $[\alpha(k), \alpha(k+1)]$. Let $P(i) = \sum_{l=0}^{i} p(l)$, $Q(j) = \sum_{l=0}^{j} q(l)$, and define recursively $s(\cdot)$ by $s(0) = 0$ and, for all $1 \leqq k \leqq m+n+1$:

$$s(k) = \min\left\{x \leqq 1 \mid x = P(i) > s(k-1) \text{ for some } i, \text{ or } x = Q(j) > s(k-1) \text{ for some } j\right\},$$

Let $v(k) \triangleq v(m(k), n(k))$. We show in Appendix that $\alpha(\cdot)$ can be define recursively by $\alpha(0) = 0$, and, for all $1 \leqq k \leqq m+n+1$:

$$\alpha(k) = \sum_{l=0}^{k-1} (s(l+1) - s(l))\, v(l).$$

Finally, denote the expected revenue by R. We prove in the Appendix that:

Theorem 5 *The expected revenue satisfies*

$$R = \sum_{l=0}^{m+n} (1 - s(l))^2 (v(l+1) - v(l)),$$

and the distribution functions used by the players are given by, for $b \in [\alpha(k), \alpha(k+1)]$,

$$\begin{aligned} p(m(k))\, F_{m(k)}(b) &= s(k+1) \frac{v(m(k), n(k)) - \alpha(k+1)}{v(m(k), n(k)) - b} - \sum_{j=0}^{m(k)} p(i), \\ q(n(k))\, G_{n(k)}(b) &= s(k+1) \frac{v(m(k), n(k)) - \alpha(k+1)}{v(m(k), n(k)) - b} - \sum_{j=0}^{n(k)} q(j). \end{aligned}$$

5 The 2x2 case

To gain further understanding into the dynamic game for arbitrary discount factors, we restrict attention in what follows to the 2x2 case, that is, $m = n = 1$. We consider first the case in which the model is symmetric, and we restrict our attention to symmetric equilibria. First it is necessary, however, to give a detailed description of the equilibrium in the one-sided subgame that can arise in the 2x2 case.

5.1 The One-Sided Case

Before developing the analysis, it may be helpful to point out one source of equilibrium multiplicity. As we have seen in the second step of the proof for the general one-sided case of the proof, the uninformed bidder does not bid $v(0) = 0$ with positive probability in any period $t < T$, where T is the maximal number of consecutive periods in which the informed bidder with a higher signal may bid zero on the equilibrium path. This follows from the observation that, as continuation play does not depend on the uninformed player's bid, bidding $\varepsilon > 0$ for ε small enough strictly dominates bidding 0, since, by bidding 0, the uninformed bidder only wins with probability $\frac{1}{2}$ an object that is worth, conditional on winning with such a bid, strictly more. This is not true in period T, in which, conditional on winning with a bid of zero, the object is worthless. Therefore, the uninformed bidder may bid zero with discrete probability in this period, and this specification is to some extent arbitrary. Of course, this affects the incentives of the high-signalled bidder to mimic the low-signalled one, and therefore the specific equilibrium strategies. For any such specification for period T, one can uniquely solve for the equilibrium strategies (in particular, derive what T is). We have therefore a continuum of equilibria, but this multiplicity vanishes as δ tends to one. To fix idea and highlight that this is the only source of multiplicity under our maintained assumption, we assume henceforth that, in period T as in the previous periods, the uninformed bidder bids zero with zero probability (conditional on observing only bids of zero so far by the informed bidder).

We normalize $v(0) = 0$, $v(1) = 1$, and let $p = p(1)$. If the informed bidder gets a low signal, he bids 0 in every period. Therefore, as soon as a strictly positive bid is observed, bids are 1 from then on ("no underbidding"). It remains to determine strategies for histories such that all bids by the informed player have been zero so far, and all notations that follow are conditioned on such a history. We let p_t be the probability that the value is 1 in period t. By definition $p_0 = p$, which we assume in $[0, 1)$ to avoid trivialities. We define β_t to be the highest bid in the support of either player, H_t to be bid distribution in period t used by the informed bidder with a high signal, and U_t to be the bid distribution in period t used by the uninformed bidder. We claim that:

1. if $1 - p_t < \delta$, then the informed player with high valuation must bid 0 with strictly positive probability. If he does not, then given his equilibrium bid, his continuation payoff is zero (because his valuation will be known) and his payoff in period t must therefore be his payoff in the static auction given beliefs p_t, that is $1 - p_t$. On the other hand, by bidding 0 instead, he will be able to win the unit in period $t + 1$ at negligible cost, as the uninformed bidder will wrongly believe that he faces an informed player with low valuation. His payoff from doing so is therefore δ, which must be lower than his equilibrium payoff, an immediate contradiction.

2. If $1 - p_t \geqq \delta$, then the informed bidder with high valuation cannot bid 0 with positive probability. If he did, his payoff (in period t) would be strictly bounded above by δ, as he can have a strictly positive reward in at most one period, this reward is strictly lower than one given that he must make a strictly positive bid to win, and this reward can come no sooner than in period $t + 1$.

3. The equilibrium sequence $\{p_t\}$ is weakly decreasing and gets below $1 - \delta$ in finite time. This follows from observations 1 and 2, Bayes' rule, the obvious fact that an informed bidder with low valuation bids zero in equilibrium, and the fact that the informed bidder with high valuation is only willing to bid zero with positive probability finitely many times. The last point follows from the fact that his payoff in the initial period is bounded below by $1 - q$, and his payoff in the initial period is bounded above by δ^t, where t is the number of initial periods during which he is willing to bid zero with positive probability.

We can now explicitly solve for the equilibrium strategies. Let T denote the first period t in which $1 - p_t \geqq \delta$.

Because the informed bidder with high valuation is willing to bid 0 before, we must have, for $t < T$,

$$\begin{aligned} 1-\beta_t &= \delta\left(1-\beta_{t+1}\right), \text{ or} \\ 1-\beta_t &= \delta^{T-t}\left(1-\beta_T\right) \end{aligned}$$

In addition, because of Bayes' rule,

$$1-p_{t+1} = \frac{1-p_t}{1-\beta_t}.$$

In period T, as the 'static' auction is played, we must have $p_T = \beta_T$, and of course $p_0 = p$. We can therefore solve for $1-p_T$.

$$1-p_T = \frac{1-p_0}{\left(1-\beta_{T-1}\right)\left(1-\beta_{T-2}\right)\cdots\left(1-\beta_0\right)} = \frac{1-p_0}{\delta\left(1-\beta_T\right)\delta^2\left(1-\beta_T\right)\cdots\delta^T\left(1-\beta_T\right)} = \frac{1-p}{\delta^{T(T+1)/2}\left(1-p_T\right)^T},$$

and thus:

$$1-p_T = (1-p)^{1/(T+1)}\delta^{-T/2}.$$

Also, $1-p_{T-1} = \delta\left(1-p_T\right)^2$. Therefore, T must satisfy $1-p_T \geqq \delta$ and $\delta\left(1-p_T\right)^2 < \delta$, i.e. $1-p_T < 1$, that is:

$$\delta^{(T+1)(T+2)/2} \leqq 1-p < \delta^{T(T+1)/2}.$$

As $\left\{\left[\delta^{(n+1)(n+2)/2}, \delta^{n(n+1)/2}\right)\right\}_{n\in\mathbb{N}}$ partition the unit interval, this defines $T = \min\left\{t \in \mathbb{N}; 1-p \geqq \delta^{(T+1)(T+2)/2}\right\}$ and establishes its existence and uniqueness. Observe that, in accordance with the results proved above more generally, $\lim_{\delta\to 1} T = \infty$, but $\lim_{\delta\to 1}\delta^T = 1$. Finally, we compute the distribution functions:

$$U_t(b) = \delta^{T-t}\frac{1-p_T}{1-b},\ p_t H_t(b) = \delta^{T-t}\frac{1-p_T}{1-b} - (1-p_t).$$

In this simple example, we can solve for the equilibrium payoff of the players, and study the expected bid trajectory. For the payoff, we get:

$$\begin{aligned} \Pi^I(p) &= (1-\delta)\,\delta^{T/2}(1-p)^{\frac{1}{T+1}}, \\ \Pi^U(p) &= (1-\delta)\,\delta^T\left[\sum_{t=1}^{T}\delta^{\frac{(t-2)(T+1-t)}{2}}(1-p)^{\frac{t}{T+1}}\right] - (1-p)\left(1-\delta^T\right), \end{aligned}$$

where $\Pi^I(p)$ and $\Pi^U(p)$ are the (initial) payoffs of the informed high-signalled player and the uninformed player, respectively, given belief p. We observe that $\Pi^I(p)$ is continuous and decreasing in p and in δ, $\Pi^I(0) = 1-\delta$, $\lim_{p\to 1}\Pi^I(p) = 0$, and $\Pi^I(p)$. As for Π^U, it is continuous and increasing in p, $\Pi^U(p) = 0$, $\lim_{p\to 1}\Pi^U(p) = 1$, but $\lim_{\delta\to 1}\Pi^U(p) = 0$.

Finally, we study the variations of the expected bids. The expected maximum bid (conditional, as usual, on the informed bidder having bid 0 up to $t-1$) is

$$E_t = \left(1-\delta^{T-t}\left(1-p_T\right)\right)^2,$$

which is decreasing in t. The *unconditional* expectation of the winning bid in period $t \geqq 1$, $t \leqq T$, F_t, is given by:

$$F_t = 1-\delta^T\left(1-p^T\right)\frac{\delta^{-t}-1}{\delta^{-1}-1}\left(1-\left(1-\delta^{T-t}\left(1-p_T\right)\right)^2\right),$$

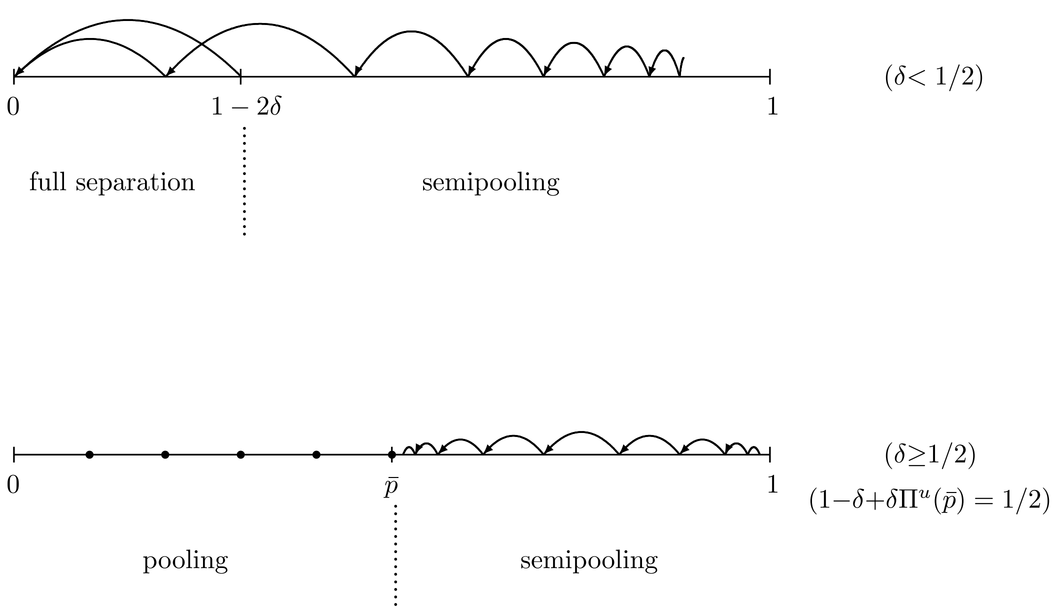
which is decreasing in t as well. Of course, for $t > T$, it is equal to the prior, p, and is larger than the corresponding expectation for all $t \leqq T$. Details for all of the calculations above can be found in the Appendix.

5.2 The Symmetric Case

Symmetry is defined as $p(j,k) = p(k,j)$, $v(j,k) = v(k,j)$, for all $j \in M$, $k \in N$. Let $p = p(S_i = 1 \mid S_{-i} = 1)$ be the probability that a player's opponent has a high signal given that the player has observed a high signal. We assume furthermore that types are independently distributed and normalize $v(1,1) = 2$, $v(0,0) = 0$ and set $v(0,1) = 1$. We have verified that the analysis can be made without independence and with arbitrary $v(0,1) \in (v(0,0), v(1,1))$, but there is very little to be gained from such generality, and notation becomes significantly more cumbersome. Let us call an equilibrium *separating* if it involves the low types of each player bidding 0 and the high types continuously randomizing over $[0,\beta]$, *semi-pooling* if low types bid a common bid p', while high types randomize on $[p',\beta]$ with an atom at p', and *pooling* if both types make a common bid p'. As mentioned before, we restrict attention to equilibria in which equilibrium strategies are symmetric (as long as information remains nondegenerate two-sided incomplete). In the subgames of one-sided incomplete information following separation by a high type (who is now the uninformed), we follow the equilibrium described in the previous subsection, but with all bids shifted upward by 1 due to 5(ii).

Theorem 6 *For all δ, the equilibrium strategies are unique. For $\delta < 1/2$, the equilibrium is separating if $1-p \geqq 2\delta$, and semi-pooling otherwise. For $\delta > 1/2$, the equilibrium is pooling if $\Pi^U(p) \leqq \delta - 1/2$, and semi-pooling otherwise.*

This theorem is illustrated in the next figure.



The diagonal $p_1 = p_2$

Proof. For a separating equilibrium to exist, the only incentive constraint to verify is that high types have no incentive to mimic low types. By deviating, a high type gets a payoff of $(1-\delta)(1-p)/2 + \delta(1-\delta)$. By abiding by the equilibrium strategy, he gets $(1-\delta)(1-p)$. Therefore, it is necessary and sufficient that $2\delta \leqq 1-p$. For a pooling equilibrium to exist, the only incentive constraint to verify is that a high type does not want to deviate and bid slightly more. By deviating, he can get up to $1-\delta+\delta\Pi^{U}(p)$, and by abiding by the equilibrium, he gets $\frac{1}{2}$. Therefore, a necessary and sufficient condition for a pooling equilibrium to exist is that $1-\delta+\delta\Pi^{U}(p)\leqq\frac{1}{2}$.

Consider now a semi-pooling equilibrium. Let γ be the probability that one's opponent is of the high type and makes a bid larger than p' (a *separating* bid), which is both the lowest bid (the *pooling* bid) and the posterior belief about the opponent's type, conditional on the pooling being observed. Then, for high types to be indifferent between the pooling bid and a slightly higher bid, we need:

$$\begin{aligned} V(p) &= \underbrace{\gamma\delta\Pi^{I}(p')+(1-\gamma)\left((1-\delta)\frac{1}{2}+\delta V(p')\right)}_{\text{payoff from bidding } p'} \qquad (5)\\ &= \underbrace{(1-\gamma)\left((1-\delta)+\delta\Pi^{U}(p')\right)}_{\text{payoff from bidding } p'_{+}}, \end{aligned}$$

where $V(p)$ is the value given belief p, and $V(p')$ is the value given p' (uniqueness will be shown), and Bayes' rule gives that

$$p'=\frac{p-\gamma}{1-\gamma}, \text{ or } 1-\gamma=\frac{1-p}{1-p'}.$$

Rearranging gives

$$\begin{aligned} 1-p &= \frac{\delta\Pi^{I}(p')}{(1-\delta)\frac{1}{2}+\delta\left(\Pi^{U}(p')+\Pi^{I}(p')-V(p')\right)}(1-p'), \qquad (6)\\ V(p) &= \frac{\delta\Pi^{I}(p')\left(1-\delta+\delta\Pi^{U}(p')\right)}{(1-\delta)\frac{1}{2}+\delta\left(\Pi^{U}(p')+\Pi^{I}(p')-V(p')\right)}. \end{aligned}$$

We show that such an equilibrium cannot exist for $\delta<1/2$ and $2\delta\leqq 1-p$ (the latter clearly implies the former), nor can it exist for $\delta\geqq 1/2$ and $1-\delta+\delta\Pi^{U}(p)\leqq\frac{1}{2}$ (same remark). Consider the first case. Observe that a (decreasing) sequence $(p,p',p'',\ldots)$ of consecutive semi-pooling equilibria cannot have an accumulation point, for by picking a term arbitrarily far in the sequence, its value $V(p)$ would be arbitrarily close to $\frac{1}{2}$, while the deviation payoff for a high type would be arbitrarily close to $1-\delta+\delta\Pi^{U}(p)>\frac{1}{2}$, a contradiction (this argument covers also the case where the alleged accumulation point is 0); therefore, any sequence of consecutive equilibria must have a largest term, after which, by necessity, the equilibrium is separating. Pick the largest such term. Equation (5) implies that

$$(p-p')\,\delta\Pi^{I}(p')+(1-p)\left(\delta(1-\delta)(1-p')\right)=(1-p)\left((1-\delta)\frac{1}{2}+\delta\Pi^{U}(p')\right).$$

Because $\Pi^{I}(p')\leqq 1-\delta$ and $\Pi^{U}(p')\geqq 0$, this requires

$$1-p'(2-p)\geqq\frac{1-p}{2\delta}.$$

However, this is impossible if $2\delta<1-p$, and only possible for $2\delta=1-p$ if $p'=0$, that is, if the equilibrium is in fact separating. Consider now the second case. Suppose first that there is a sequence $(p,p',p'',\ldots)$ with infinitely many terms that are semi-pooling. Then we can pick terms such that the payoff must be arbitrarily close to $1/2$ and $1-\delta+\delta\Pi^{U}(p)$, implying that the limit of such a sequence cannot be such that $1-\delta+\delta\Pi^{U}(p)<\frac{1}{2}$. We can then suppose without loss of generality that, if a semi-pooling exists with $1-\delta+\delta\Pi^{U}(p)<\frac{1}{2}$, then (conditional on both players making a pooling bid), players pool (forever) at belief p'. Equation (5) then implies that

$$(p-p')\,\delta\Pi^{I}(p')+(1-p)\frac{1}{2}=(1-p)\left(1-\delta+\delta\Pi^{U}(p')\right), \qquad (5')$$

which is impossible because $1-\delta+\delta\Pi^{U}(p')<1-\delta+\delta\Pi^{U}(p)<\frac{1}{2}$, and $\Pi^{I}(p')\geqq 0$. In fact, the same argument rules out a semi-pooling equilibrium with p such that $1-\delta+\delta\Pi^{U}(p)=\frac{1}{2}$. This establishes the second claim. It may be worthwhile at this point to observe that, if semi-pooling equilibria exists for $1-\delta+\delta\Pi^{U}(p)>\frac{1}{2}$, then the sequence of continuation plays (conditional on players making the pooling bid) is an infinite sequence consisting exclusively of semi-pooling equilibria. If there is a limit, it must obviously satisfy $1-\delta+\delta\Pi^{U}(p)=\frac{1}{2}$, so this limit is unique. As pooling equilibria cannot exist for $1-\delta+\delta\Pi^{U}(p)>\frac{1}{2}$, all we have to show is that such a sequence cannot be finite, i.e. such that there exists p, after which (conditional on both players making pooling bids) the equilibrium is pooling (i.e., p' is such that $1-\delta+\delta\Pi^{U}(p')\leqq\frac{1}{2}$). This immediately follows from (5').

We are left with establishing existence (and uniqueness) of semi-pooling equilibria for $\delta<1/2$ and $2\delta>1-p$, as well as for $\delta>1/2$ and $1-\delta+\delta\Pi^{U}(p)>\frac{1}{2}$. As shown, if such a semi-pooling equilibrium exists, then it must belong to a sequence of semi-pooling equilibria which is finite in the first case, ending up with a separating equilibrium, and infinite in the second case, converging to the unique p solving $1-\delta+\delta\Pi^{U}(p)=\frac{1}{2}$.

Consider first $\delta<1/2$ and $2\delta>1-p$. We show that (6) determines uniquely $(V(p),p)$ as a function of $(V(p'),p')$, and that the (first coordinate of the) sets of consecutive solutions of (6), with boundary conditions given by $(p,(1-\delta)(1-p))$ for $p\in(0,1-2\delta]$ partition $(1-2\delta,1)$. To see this, suppose that $V(p')$ is decreasing in p' and smaller than $(1-\delta)$. It follows from the first equation of (6) that $(1-p)/(1-p')$ is decreasing in p'. This follows from the fact that

$$\frac{(1-\delta)\frac{1}{2}+\delta\left(\Pi^{U}(p')+\Pi^{I}(p')-V(p')\right)}{\delta\Pi^{I}(p')}=1+\frac{\Pi^{U}(p')}{\Pi^{I}(p')}+\frac{(1-\delta)\frac{1}{2}-\delta V(p')}{\delta\Pi^{I}(p')}$$

is increasing (Π^{U} and $1/\Pi^{I}$ is increasing and positive, and so is $(1-\delta)\frac{1}{2}-\delta V(p')$, because $\frac{1}{2}>\delta$, and, by hypothesis, $(1-\delta)\geqq V(p')$, and $V(p')$ is decreasing. The fact that is $(1-p)/(1-p')$ is decreasing in p' implies in particular that p increases with p', but also, from (5'), since

$$V(p)=\left(1-\frac{1-p}{1-p'}\right)\delta\Pi^{I}(p')+\frac{1-p}{1-p'}\left((1-\delta)\frac{1}{2}+\delta V(p')\right),$$

as $\delta\Pi^{I}(p')\leqq(1-\delta)\frac{1}{2}\leqq(1-\delta)\frac{1}{2}+\delta V(p')$, and both $\delta\Pi^{I}(p')$ and $(1-\delta)\frac{1}{2}+\delta V(p')$ are decreasing in p', that $V(p)$ decreases with p' (as a weighted average of two decreasing functions with increasing weight on the smaller one).Because $V(p)$ is decreasing, it follows in particular that $V(p)\leqq 1-\delta$, provided that V is continuous, which follows by induction as well once it is established for the first iteration. It is immediate to verify that for $p'=\varepsilon>0$ arbitrarily small, and associated $V(p')=(1-\delta)(1-p')$, there exists $(p,V(p))$ solving (6) arbitrarily close to $(1-2\delta,2\delta(1-\delta))$. Because (the projection on the first coordinate space of) the image by (6) of the interval $(0,1-2\delta]$ is an interval $(1-2\delta,p^{*}]$, for some $p^{*}>1-2\delta$, and the value V is continuous on that interval, it follows by induction that the intervals of probabilities constructed this way have neither "gaps" nor "overlaps", and that V is continuously decreasing in p. Observe that, as $p'=1$ is a fixed point of the first equation of (6), the union of these intervals never stretches above one. Conversely, because, for $p'\geqq 2\delta(1-\delta)$ (which is certainly a probability that is "reached", since it belongs to the "first" interval),

$$1-p=\frac{\delta\Pi^{I}(p')}{(1-\delta)\frac{1}{2}+\delta\left(\Pi^{U}(p')+\Pi^{I}(p')-V(p')\right)}(1-p')<\frac{1-p'}{1+(1-\delta)(1-2\delta)},$$

for any $p<1$ is eventually included in the union of intervals recursively obtained by application of (6). This proves that for every $p>1-2\delta$, there exists one and only one equilibrium outcome, specifying in particular that players semi-pool as long as they have pooled, until the common belief p is less than $1-2\delta$, at which point separation occurs.

Let us now study the case $\delta \geqq 1/2$ and $1-\delta+\delta\Pi^U(p) > \frac{1}{2}$. Defining $q = 1/(1-p)$, $w = V(p)/(1-p)$, and $f(q) = 1-\delta+\delta\Pi^U(1-1/q)$, we get from (6) the following pair of difference equations

$$\begin{cases} q_{n+1} - q_n = \frac{\left(f(q_n)-\frac{1-\delta}{2}\right)q_n - \delta w_n}{\delta\Pi^I(q_n)} \\ w_{n+1} = f(q_n), \end{cases}$$

where calendar time is reversed, that is, q_n corresponds to the posterior belief given semi-pooling and a prior belief q_{n+1}. Observe that the unique critical point of this system is $\bar{x} \triangleq (\bar{q}, \bar{w}) = \left((1-\bar{p})^{-1}, w(\bar{p})\right)$, where $\bar{p}$ is the unique root of $1-\delta+\delta\Pi^U(p) = \frac{1}{2}$. For later use, observe also that this system is equivalent to the second-order difference equation

$$q_{n+1} - q_n = \frac{(f(q_n)-(1-\delta)/2)\,q_n - \delta f(q_{n-1})}{\delta\Pi^I(q_n)},$$

from which it is apparent that $q_n > q_{n-1} > \bar{q}$ implies $q_{n+1} > q_n > \bar{q}$ (since then $(f(q_n)-(1-\delta)/2)\,q_n - \delta f(q_{n-1}) > (1-\delta)\left(f(q_n)-\frac{1}{2}\right)$). Computing the Jacobian evaluated at this fixed point, we get:

$$\begin{bmatrix} 1 + \frac{f'(\bar{q})\bar{q}+f(\bar{q})-(1-\delta)/2}{\delta\Pi^I(\bar{q})} & -\frac{1}{\Pi^I(\bar{q})} \\ f'(\bar{q}) & 0 \end{bmatrix},$$

whose roots are real conjugate, one of which has modulus strictly less than one, the other one has modulus strictly larger than one. Indeed, the discriminant is positive because $\left(1+\frac{f'(\bar{q})\bar{q}+f(\bar{q})-(1-\delta)/2}{\delta\Pi^I(\bar{q})}\right)^2 > 4f'(\bar{q})/\Pi^I(\bar{q})$, as the term which is squared exceeds $\left(1+f'(\bar{q})/\Pi^I(\bar{q})\right)^2$ (using $f(\bar{q})-(1-\delta)/2 > 0$ and $\bar{q}/\delta > 1$), and the ordering of moduli is easily established using the same bounds. Therefore, $\bar{x}$ is a hyperbolic fixed point, the map defined by the system of difference equations has a saddle point at $\bar{x}$, and so has its inverse map (the eigenvalues of the inverse matrix are the inverses of the eigenvalues). By the stable manifold theorem (see Devaney (1989)), there exists a neighborhood of $\bar{x}$ such that, for each q in this neighborhood, there exists a unique w such that the limit of the system starting from (q, w) is $\bar{x}$. Because $q_n = (1-p_n)^{-1}$ is strictly increasing in p_n, and $w_n = f(q_{n-1})$ is similarly increasing in p_{n-1}, we may therefore conclude that, using standard calendar time, there exists a neigborhood of $\bar{p}$, such that, for each p_n in this neighborhood, there exists a unique p_{n+1} in this neighborhood, such that the sequence (p_k) going consecutively through p_n and p_{n+1} tends to $\bar{p}$. Evidently, $p_{n+1} > \bar{p}$. As p_{-n} is monotonic since q_{-n} is, and $\bar{p}$ is the unique fixed point of the second-order difference equation, it follows that through all points $p \in (\bar{p}, 1)$ such a sequence exists, and uniqueness follows from trivial continuity and monotonicity observations. ■

5.3 The Asymmetric Case

In this section we extend the case of the diagonal ($p_1 = p_2 = p$) to the asymmetric version; WLOG assume $p_2 > p_1$. For $\delta < 1/2$ (available from the authors), bidding is fairly complex. Typically, both types of both players bid on overlapping but non-nested supports. However, we can say that there is continuity with both the diagonal and the static game, in that as δ goes to 0 the equilibrium play converges to the equilibrium of the static case. The general version of the static case was described in section 4.3, but for concreteness we spell it out now for binary types and independent signals.

Assuming that $p_2 \geqq p_1$ and $\delta = 0$, denote the c.d.f. of player i by H_i or L_i, depending upon whether his valuation (i.e. signal) is high or low. L_2 is degenerate with unit mass at 0, H_2 has support $[0, p_1+p_2]$ (with no atom at 0), H_1 has support $\left[\frac{p_2-p_1}{1-p_1}, p_1+p_2\right]$, and L_1 has support $\left[0, \frac{p_2-p_1}{1-p_1}\right]$ with an atom at 0. Low types have zero payoff, while player i's high type payoff is $1-p_i$. There is, of course, full separation of types. The specific distributions are

$$\begin{aligned}
L_1(b) &= \frac{1-p_2}{(1-p_1)(1-b)},\ b \leqq \frac{p_2-p_1}{1-p_1}, \\
H_2(b) &= \frac{1-p_2}{p_2}\frac{b}{1-b},\ b \leqq \frac{p_2-p_1}{1-p_1}, \\
H_i(b) &= \frac{1-p_{-i}-(1-p_i)(1-b)}{p_i(2-b)},\ b \in \left[\frac{p_2-p_1}{1-p_1}, p_1+p_2\right].
\end{aligned}$$

Thus in the symmetric case $p_1 = p_2 = p$, both low types bid 0 for sure, and both high types bid in the range $[0, 2p]$ and make profits of $1-p$. The expected revenue for the auctioneer, in the general case, is $p_1^2 + p_2^2$.

We study the $\delta \geqq 1/2$ case in more detail here. For existence, we introduce an endogenous tie-breaking rule. Recall that we denote $1-\delta+\delta\Pi^U(p_i)$ by $f(p_i)$, with $\bar{p}$ satisfying $f(\bar{p}) = \frac{1}{2}$. Then if the bid at which the players tie is at least $\bar{p}$, they win with equal probability as before. But if their tying bid b is less than $\bar{p}$, we give player 2 a share $f(b) < \frac{1}{2}$ and we give player 1 the remaining share $1-f(b)$. Given that in equilibrium the pooling bid will be at the posterior beliefs about player 1, this boils down to giving player 2 exactly his outside option. However, the rules are of course independent of the players' beliefs and can be implemented with observable bids only.

First note that the edge $p_1 = 0$ corresponds to the one-sided case with degenerate beliefs on the low type of player 1. Here the equilibrium is again pooling (for any p_2): both players bid 0 and receive the object with probability δ and $1-\delta$ respectively, as above. The high type of player 2 weakly prefers his equilibrium payoff, namely $f(0) = 1-\delta$, to his payoff from deviation, which is also $1-\delta$ ($= (1-\delta)\cdot 1 + \delta \cdot 0$). Similarly, the uninformed player 1 (low type) prefers δp_2 to $(1-\delta)p_2$ as long as $\delta \geqq 1/2$. If the high type of player 1 were around (which he isn't!), he would be willing to pool exactly when

$$\begin{aligned}
\delta(1+p_2) &\geqq (1-\delta)(1+p_2) + \delta\Pi^U(p_2), \text{ or} \\
(2\delta-1)p_2 + \delta &\geqq f(p_2).
\end{aligned}$$

For $\delta < 2/3$, this yields an upper bound $\bar{p}_2 \geqq \bar{p}$ (with strict inequality as long as δ is strictly larger then 1/2); for $\delta \geqq 2/3$, it is satisfied for any p_2. This bound is not crucial at this point, since there is no incentive constraint for a type who does not exist, but it will arise again below. However, it is the very possibility of a higher type of player 1 that allows us to support a pooling equilibrium in the first place: the difference between knowledge and belief.

The equilibrium we consider for the two-sided case (we have not proven uniqueness) involves pooling below a boundary connecting the diagonal (i.e. the symmetric case) to either the edge that corresponds to $p_1 = 0$ (for $\delta < 2/3$) or to the edge corresponding to $p_2 = 1$ (for $\delta \geqq 2/3$). This boundary intersects the diagonal at the point $p = \bar{p}$, so this equilibrium truly does extend that of the symmetric case. It intersects the edge $p_1 = 0$ (for $\delta < 2/3$) exactly at the point $\bar{p}_2$ defined above, and it is monotonic in p_1 (for any δ). It is described by the following expression:

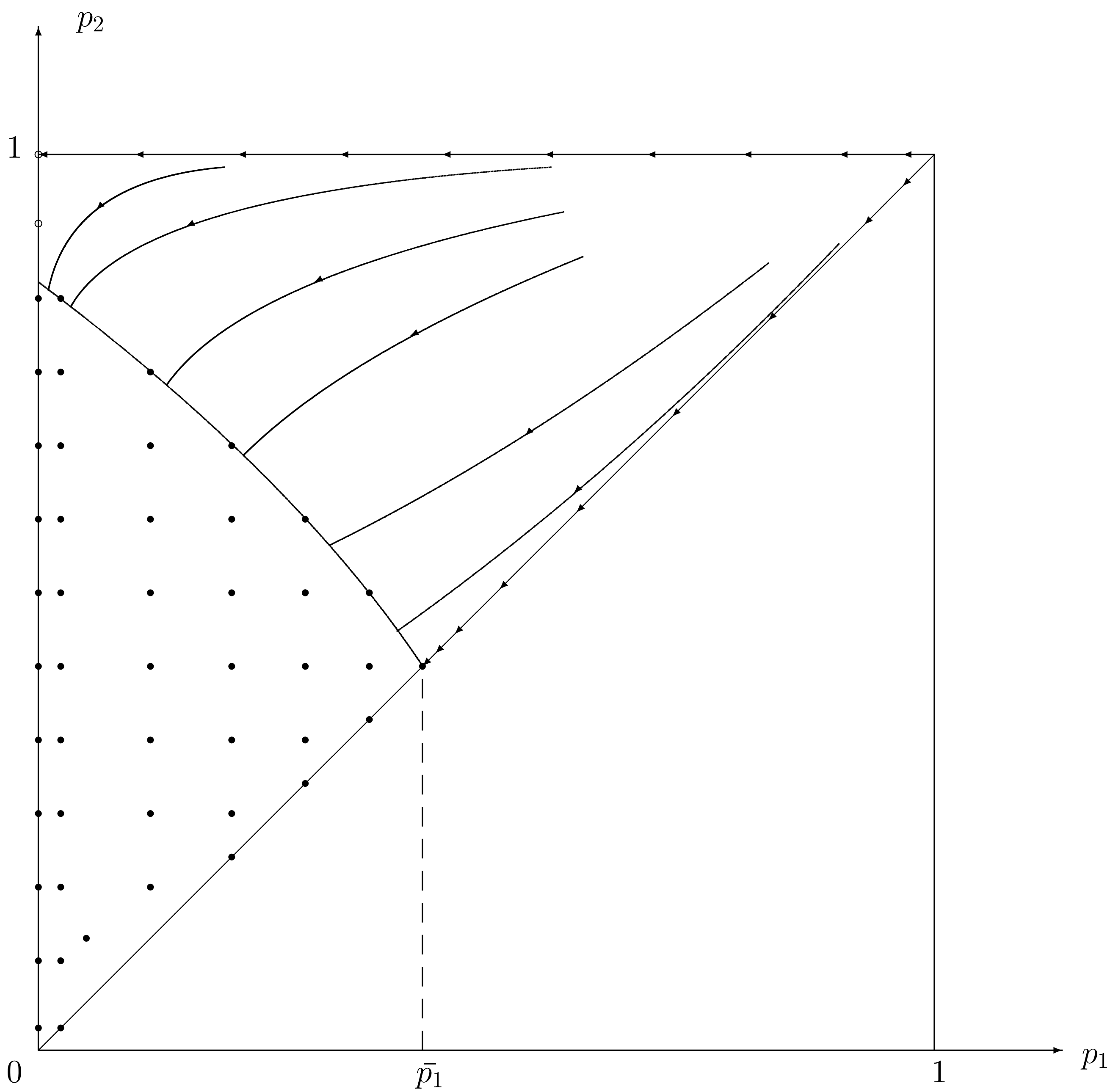
$$1 + (\delta - f(p_1))(p_2 - p_1) = f(p_1) + f(p_2).$$

Above this boundary line, the players semipool, where the low pooling bid is the posterior belief on player 1's type. That is, both high types pool with positive probability and also make revealing bids with positive probability. They converge toward the pooling region (conditional on no one having yet separated), but do not reach it in finite time, just as in the symmetric case. The specific equations for the evolution of beliefs under semipooling are given in the Appendix. Of course, given Theorems 2 and 3, as δ approaches 1 the boundary line expands and the pooling region fills the entire parameter space. In this case the seller's [normalized] expected revenue is the common pooling bid p_1, which is certainly lower than the overall expected value of $p_1 + p_2$, and may or may not be lower than the revenue in the static game, $p_1^2 + p_2^2$. On the other hand, as δ approaches 1/2, the pooling region shrinks and we have only semipooling in the limit.

We formalize these results in the following theorem, which is proved in the Appendix.

Theorem 7 *For $\delta \geqq 1/2$ and any $p_2 > p_1 > 0$, there is an equilibrium satisfying all our assumptions; it is characterized by pooling if $1 + (\delta - f(p_1))(p_2 - p_1) \geqq f(p_1) + f(p_2)$ and by semipooling otherwise.*

These results are summarized in the following Figure.



$1-\delta+\delta\Pi^u(\bar{p_1}) = 1/2$

(Conditional) evolution of (p_1,p_2) (given $\delta \geq 1/2$)

6 Concluding Comments

This paper shows that in an infinitely repeated, first-price, common-value auction in which the value of all units is perfectly correlated, rational and patient players prefer to make a low, "pooling" bid in every period rather than divulge any of their proprietary information. We have imposed four refinements in order to prove uniqueness of this equilibrium outcome. First, stationarity: players bids depend only on beliefs. This assumption is necessary to eliminate "reputational" equilibria as described in the durable goods monopoly literature (Ausubel and Deneckere (1989)). Consider for instance the one-sided version of our model (Section 3). By Theorem 1, as the discount factor tends to one, both the uninformed and the informed bidder's payoff tend to zero. Then, without stationarity, there exists a P.B.E. (satisfying the other three refinements) in which both players repeatedly bid $v(0)$ and win the object with equal probability. The informed bidder has no incentive to deviate if any higher bid is interpreted as evidence of, say, him being the highest possible type, and the uninformed bidder has no incentive to deviate either if such a bid triggers a reversion to any equilibrium satisfying Theorem 1. In turn, this makes it possible to construct other equilibria of the repeated game with two-sided incomplete information. Second, it is necessary to assume "no underbidding", that is, that players bid at least as much as the object is commonly believed to be worth. Otherwise, there exists a continuum of other pooling equilibrium which differ only in the pooling bid, and our refinement selects the equilibrium with the highest such bid. Third, we assumed that players' revise their belief in such a way as to assign zero probability to a player's type for which the observed bid is more than what the object could conceivably be worth to him (that is, under his most optimistic conjectures). As mentioned, we suspect that this third refinement could be dispensed with. Finally, we assumed in section 3.2 that players (as a union across types) use connected bidding supports. Without this assumption, there is, for instance, an equilibrium of the one-sided degenerate-belief version of our game in which the informed player always bids λ (his lowest possible value for the object), and the uninformed player randomizes between λ and precisely the one bid $\beta > \lambda$ so that winning with probability 1 at β gives him the same static payoff as winning half the time at λ (because he would tie). If all out-of-equilibrium bids are interpreted as coming from the highest type and thus leading to the outcome of Theorem 1, no type of any player has an incentive to deviate. We find this equilibrium somewhat artificial; connectedness rules it out.

How could an auctioneer eliminate the tacit collusion that our equilibrium suggests? A reserve price is an instrument that, used wisely, would allow the auctioneer to fare better. If he can commit to a reserve price policy, then as the discount factor tends to one, his optimal expected revenue tends toward his revenue from setting an optimal fixed reserve price. Players whose signal is sufficiently high pool at a level slightly above the reserve price, while players with lower signals remain idle. Therefore, the auctioneer's expected revenue is still lower than in the static auction. Another commonly used procedure in auctioneering is the option to a winner to purchase future units at the current price (See Cassady (1967)). It is clear that such a procedure eliminates the pooling equilibrium, as at least one player would have an incentive to bid a penny more and exercise his option. However, this procedure is not perfect either, as a player with a low signal may exercise his option and win all units at a price below their real value (assuming that bids are observed before the option decision is made).

As in Kyle and the literature on insider trading, we have restricted attention to the case in which the values of the units are perfectly correlated. This seems to be the most challenging case for information revelation. Indeed, suppose that the value of each unit is an independent draw from some (possibly time-dependent) distribution, for which the static first-price auction admits a unique equilibrium. Then the only equilibrium that is stationary in the repeated game specifies that the static auction be played in each period.

We have also restricted attention to the two player case. We have verified that the pooling equilibrium remains an equilibrium satisfying our refinements when there are more than two players, but have not proved uniqueness. As soon as a single player has revealed his information, all other players' private information must be eventually revealed, as such an uninformed bidder cannot be disciplined into pooling (given stationarity) and thus, informed bidders must eventually act. Such information revelation occurs "quickly" relative to the discount factor, which in turn allows players to enforce tacit collusion when none of them has revealed any information. The intuition for uniqueness appears robust: if none of the opponents reveals his private information, then pooling is certainly optimal provided the discount factor is high enough, while if some of them do reveal theirs, it is then still better to pool and become the informed player, for uninformed bidders have a zero payoff as soon as there is more than one of them.

We have also considered the framework of private values (rather than common) and of second-price auctions (rather than first-price). However, it is clear that much work remains to be done in the domain of repeated auctions.

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7 Appendix

7.1 Static case

Proof of Theorem 5: Because player 1's type $m(k)$ is indifferent between bidding $\alpha(k)$ and $\alpha(k+1)$,

$$\left(\sum_{j=0}^{n(k)-1} q(j) + q(n(k)) G_{n(k)}(\alpha(k))\right)(\alpha(k+1)-\alpha(k)) = q(n(k))\left(G_{n(k)}(\alpha(k+1)) - G_{n(k)}(\alpha(k))\right) v(m(k), n(k)),$$

and, similarly, because player 2's type $n(k)$ is indifferent between bidding $\alpha(k)$ and $\alpha(k+1)$,

$$\left(\sum_{i=0}^{m(k)-1} p(i) + p(m(k)) F_{m(k)}(\alpha(k))\right)(\alpha(k+1)-\alpha(k)) = p(m(k))\left(F_{m(k)}(\alpha(k+1)) - F_{m(k)}(\alpha(k))\right) v(m(k), n(k)).$$

Therefore, upon dividing,

$$\frac{q(n(k))\left(G_{n(k)}(\alpha(k+1)) - G_{n(k)}(\alpha(k))\right)}{\sum_{j=0}^{n(k)-1} q(j) + q(n(k)) G_{n(k)}(\alpha(k))} = \frac{p(m(k))\left(F_{m(k)}(\alpha(k+1)) - F_{m(k)}(\alpha(k))\right)}{\sum_{i=0}^{m(k)-1} p(i) + p(m(k)) F_{m(k)}(\alpha(k))}.$$

For $k \leqq m+n+1$, define $x(k) = \sum_{i=0}^{m(k)-1} p(i) + p(m(k)) F_{m(k)}(\alpha(k))$ and $y(k) = \sum_{i=0}^{n(k)-1} q(j) + q(n(k)) G_{n(k)}(\alpha(k))$. It follows that:

$$\frac{x(k+1)-x(k)}{x(k)} = \frac{y(k+1)-y(k)}{y(k)},$$

and thus, because $x(m+n+1) = y(m+n+1) = 1$, $x(k) = y(k)$, for all $k \leqq m+n+1$. Because for all $\alpha(k+1)$, either $G_{n(k)}(\alpha(k+1))$ or $F_{m(k)}(\alpha(k+1))$ equals 1, this establishes that $s(k) = x(k)$ can be defined recursively as: $s(0) = 0$, and, for $P(i) = \sum_{l=0}^{i} p(l)$, $Q(j) = \sum_{l=0}^{j} q(l)$,

$$s(k) = \min\left\{x \leqq 1 \mid x = P(i) > s(k-1) \text{ for some } i, \text{ or } x = Q(j) > s(k-1) \text{ for some } j\right\},$$

for all $1 \leqq k \leqq m+n+1$. Obviously, $s(m+n+1) = 1$.

Consider again the indifference of player 2's type $n(k)$ between bids $\alpha(k)$ and $\alpha(k+1)$:

$$\begin{aligned} & \sum_{i=0}^{m(k)} p(i)(v(i, n(k)) - \alpha(k)) + p(m(k)) F_{m(k)}(\alpha(k))(v(m(k), n(k)) - \alpha(k)) \\ = & \sum_{i=0}^{m(k)} p(i)(v(i, n(k)) - \alpha(k+1)) + p(m(k)) F_{m(k)}(\alpha(k+1))(v(m(k), n(k)) - \alpha(k+1)). \end{aligned}$$

It follows that, for all $0 \leqq k \leqq m+n$:

$$\frac{s(k)}{s(k+1)} = \frac{v(m(k), n(k)) - \alpha(k+1)}{v(m(k), n(k)) - \alpha(k)},$$

and therefore, $\alpha(0) = 0$, and, for all $1 \leqq k \leqq m+n+1$:

$$\alpha(k) = \sum_{l=0}^{k-1} (s(l+1) - s(l)) v(m(l), n(l)).$$

In the same way, we determine the distribution functions. If $b \in [\alpha(k), \alpha(k+1)]$,

$$\begin{aligned} p(m(k)) F_{m(k)}(b) &= s(k+1) \frac{v(m(k), n(k)) - \alpha(k+1)}{v(m(k), n(k)) - b} - \sum_{j=0}^{m(k)} p(i), \\ q(n(k)) G_{n(k)}(b) &= s(k+1) \frac{v(m(k), n(k)) - \alpha(k+1)}{v(m(k), n(k)) - b} - \sum_{j=0}^{n(k)} q(j). \end{aligned}$$

We can now compute the expected revenue R. For $0 \leqq k \leqq m+n$, let $R(\alpha(k), \alpha(k+1))$ be the expected revenue from bids $b \in [\alpha(k), \alpha(k+1)]$. Since $\sum_{j=0}^{n(k)} q(j) + q(n(k)) G_{n(k)}(b) = \sum_{j=0}^{m(k)} p(i) + p(m(k)) F_{m(k)}(b)$, it follows that

$$\begin{aligned} R(\alpha(k), \alpha(k+1)) &= s(k+1)^2 \left(\frac{v(k) - \alpha(k+1)}{v(k) - b} \right) (2b - v(k)) \Big|_{\alpha(k)}^{\alpha(k+1)} \\ &= s(k+1)^2 (2\alpha(k+1) - v(k)) - s(k)^2 (2\alpha(k) - v(k)), \end{aligned}$$

where for simplicity, $v(k) = v(m(k), n(k))$. It follows that

$$\begin{aligned} R &= \sum_{k=0}^{m+n} R(\alpha(k), \alpha(k+1)) \\ &= 2\alpha(m+n+1) - \sum_{l=0}^{m+n} \left(s(l+1)^2 - s(l)^2 \right) v(l) \\ &= \sum_{l=0}^{m+n} 2(s(l+1) - s(l)) v(l) - \sum_{l=0}^{m+n} \left(s(l+1)^2 - s(l)^2 \right) v(l) \\ &= v(m+n+1) - \sum_{l=0}^{m+n} s(l)(2 - s(l))(v(l+1) - v(l)) \quad \text{(summation by parts)} \\ &= \sum_{l=0}^{m+n} (1 - s(l))^2 (v(l+1) - v(l)). \quad \blacksquare \end{aligned}$$

7.2 Calculations for binary one-sided

Additional details for the 2x1 case: Define $\Pi^I(p)$ and $\Pi^U(p)$ to be the expected payoff of the informed and uninformed bidder, respectively, given belief p. Obviously, $\Pi^I(p) = (1-\delta)\delta^T(1-p_T) = (1-\delta)\delta^{T/2}(1-p)^{\frac{1}{T+1}}$, with $T = \min\left\{t \in N; 1 - p \geqq \delta^{(T+1)(T+2)/2}\right\}$. It is simple to verify that $\Pi^I(p)$ is decreasing in p, $\lim_{p \to 0} \Pi^I(p) = 1 - \delta$, $\lim_{p \to 1} \Pi^I(p) = 0$, and $\Pi^I(p)$ is decreasing in δ. Observe that $1 - p_t = \left(\delta^{-t/2}(1-p)^{\frac{1}{T+1}}\right)^{T+1-t}$, and, denoting the odds ratio $p_t/(1-p_t)$ by l_t, we have $(p_t - \beta_t)/p_t = l_{t+1}/l_t$. By bidding 0_+ repeatedly, the uninformed player gets:

$$\begin{aligned} \Pi^U(p) &= (1-\delta) p \left[\frac{p_0 - \beta_0}{p_0} \left(1 + \delta \frac{p_1 - \beta_1}{p_1} \left(1 + \delta \frac{p_2 - \beta_2}{p_2} \left(\cdots + \delta \frac{p_{T-1} - \beta_{T-1}}{p_{T-1}} \right) \right) \right) \right] \\ &= (1-\delta) p \left[\frac{l_1}{l_0} + \delta \frac{l_2}{l_0} + \cdots + \delta^{T-1} \frac{l_T}{l_0} \right] = (1-\delta)\delta^T \left[\sum_{t=1}^{T} \delta^{\frac{(t-2)(T+1-t)}{2}} (1-p)^{\frac{t}{T+1}} \right] - (1-p)\left(1 - \delta^T\right). \end{aligned}$$

Observe that, because $T = \min\left\{t \in N; 1-p \geqq \delta^{(T+1)(T+2)/2}\right\}$, $\delta^T \to 1$ as $\delta \to 1$. Notice also that

$$\Pi^U(p) = (1-\delta)\, p \left[\frac{l_1}{l_0} + \delta \frac{l_2}{l_0} + \cdots + \delta^{T-1}\frac{l_T}{l_0}\right] \leqq p\left(1-\delta^T\right),$$

so that $\Pi^U(p) \to 0$ as $\delta \to 1$.

Finally, we study the variations of the expected bids. Let h_t, u_t be the densities correspondingly respectively to the distributions H_t and U_t. The density of the maximum bid, m_t, is given by

$$m_t(b) = u(t)\left(1-p_t+p_tH_t(b)\right) + p_t h_t(b)\, U_t(b) = \frac{2\delta^{2(T-t)}\left(1-p_T\right)^2}{(1-b)^3},$$

and its expectation (conditional, as usual, on the informed bidder having bid 0 up to $t-1$) is

$$\begin{aligned} E_t &= 2\delta^{2(T-t)}\left(1-p_T\right)^2 \int_0^{1-\delta^{T-t}(1-p_T)} \frac{tdt}{(1-t)^3} \\ &= 2\delta^{2(T-t)}\left(1-p_T\right)^2 \left(\frac{1}{2} + \frac{1-2\delta^{T-t}\left(1-p_T\right)}{2\left(\delta^{T-t}\left(1-p_T\right)\right)^2}\right) \\ &= \left(1-\delta^{T-t}\left(1-p_T\right)\right)^2, \end{aligned}$$

which is decreasing in t. Next, observe that the *unconditional* expectation of the winning bid in period $t \geqq 1$, $t \leqq T$, F_t, satisfies

$$\begin{aligned} F_t &= 1 - \prod_{i=0}^{t-1}\left(1-\beta_i\right)\left(1-E_t\right) \\ &= 1-\left(1-p_T\right)\left(1-E_t\right)\sum_{i=0}^{t-1}\delta^{T-i} \\ &= 1-\delta^T\left(1-p^T\right)\frac{\delta^{-t}-1}{\delta^{-1}-1}\left(1-\left(1-\delta^{T-t}\left(1-p_T\right)\right)^2\right), \end{aligned}$$

which is decreasing in t as well. Of course, for $t > T$, it is equal to the prior, p, and is larger than the corresponding expectation for all $t \leqq T$.

7.3 Asymmetric binary types

Proof of Theorem 7: If both players fully pool, the common bid will be $\lambda = p_1$ (using the refinements and the fact that $p_1 < p_2$). Anyone who bids even λ_+ will be considered to be a high type (by fiat, but this is in line with 5(iii)). Since $p_1 \leqq \bar{p}$ throughout the pooling region, we use the endogenous sharing rule. The low type of player 2 makes zero profit and can do no better. The low type of player 1 makes profits of $(1-f(p_1))(p_2) > p_2/2$ and by deviating can obtain $(1-\delta)(p_2) + \delta \cdot 0$, so he will never wish to deviate. [Recall that $f(p) = 1-\delta+\delta\Pi^U(p)$ and is equal to 1/2 at $\bar{p}$.] The incentive constraint for the high type of player 2 is

$$f(p_1)(1+p_1-p_1) \geqq (1-\delta)(1+p_1-p_1) + \delta\Pi^U(p_1) = f(p_1)$$

which is always just satisfied; this is how the tie-breaking rule was constructed. For player 1, we need

$$(1 - f(p_1))(1 + p_2 - p_1) \geqq (1 - \delta)(1 + p_2 - p_1) + \delta\Pi^U(p_2).$$

This determines the boundary of the region where full pooling is an equilibrium. Rearranging, the equation for the boundary is given by

$$1 + (\delta - f(p_1))(p_2 - p_1) = f(p_1) + f(p_2).$$

Note that $p_1 = p_2 = \bar{p}$ satisfies this equality, so the boundary hits the diagonal exactly at $\bar{p}$, as desired. Furthermore, this boundary intersects the edge case $p_1 = 0$ (where $f(p_1) = 1 - \delta$) when

$$(2\delta - 1)p_2 + \delta = f(p_2),$$

i.e. at $\bar{p}_2$ (as defined in section 5.3 for $\delta < 2/3$). This makes intuitive sense, since that was exactly where the high type of player 1 would have no longer been willing to pool. Finally, fixing p_1 and p_2 less than 1, it is clear that the RHS of player 1's original incentive constraint goes to 0 as δ goes to 1. So in the limit for arbitrarily patient players, we have full pooling for any initial parameter values.

If p_1 is too large (given some p_2), so that we are outside the boundary of the pooling region, then both players semipool. Renormalize time as time-to-go until we stop semipooling (conditional on both players pooling), i.e. until one player either fully separates or fully pools. Recall that semipooling entails players with low valuations making a low, pooling bid λ (equal to $p_{1,t-1}$), while high types make the same pooling bid with positive probability, and with positive probability they continuously randomize over some support $[\lambda_+, \beta_t]$. Let $V_i(p_{1,t}, p_{2,t})$ denote the value for player i with high valuation, given the initial beliefs. Let γ_i be the probability that player $j = 3 - i$ faces a player i making a separating bid. By Bayes' rule:

$$p_{i,t-1} = \frac{p_{i,t} - \gamma_{i,t}}{1 - \gamma_{i,t}} \text{ or } \gamma_{i,t} = \frac{p_{i,t} - p_{i,t-1}}{1 - p_{i,t-1}}.$$

Because player 1 is indifferent between pooling and separating (the first equality corresponds to a pooling bid, the second to the lowest separating bid):

$$\begin{aligned}
& V_1(p_{1,t}, p_{2,t}) \\
= \; & \gamma_{2,t}\left((1-\delta)\cdot 0 + \delta\Pi^I(p_{1,t-1})\right) + \left(1 - \gamma_{2,t}\right)\left((1-\delta)\cdot(1 - f(p_{1,t-1}))(1 + p_{2,t-1} - p_{1,t-1}) + \delta V_1(p_{1,t-1}, p_{2,t-1})\right) \\
= \; & \gamma_{2,t}\left((1-\delta)\cdot 0 + \delta\cdot 0\right) + \left(1 - \gamma_{2,t}\right)\left((1-\delta)(1 + p_{2,t-1} - p_{1,t-1}) + \delta\Pi^U(p_{2,t-1})\right).
\end{aligned}$$

That is,

$$\begin{aligned}
V_1(p_{1,t}, p_{2,t}) &= \frac{1 - p_{2,t}}{1 - p_{2,t-1}}\left((1-\delta)(1 + p_{2,t-1} - p_{1,t-1}) + \delta\Pi^U(p_{2,t-1})\right) \\
\frac{p_{2,t} - p_{2,t-1}}{1 - p_{2,t}}\delta\Pi^I(p_{1,t-1}) &= f(p_{1,t-1})(1-\delta)(1 + p_{2,t-1} - p_{1,t-1}) + \delta\left(\Pi^U(p_{2,t-1}) - V_1(p_{1,t-1}, p_{2,t-1})\right).
\end{aligned}$$

Similarly, player 2 is indifferent between pooling and separating, so that:

$$\begin{aligned}
& V_2(p_{1,t}, p_{2,t}) \\
= \; & \gamma_{1,t}\left((1-\delta)\cdot 0 + \delta\Pi^I(p_{2,t-1})\right) + \left(1 - \gamma_{1,t}\right)\left((1-\delta)\cdot f(p_{1,t-1}) + \delta V_2(p_{1,t-1}, p_{2,t-1})\right) \\
= \; & \gamma_{1,t}\left((1-\delta)\cdot 0 + \delta\cdot 0\right) + \left(1 - \gamma_{1,t}\right)\left((1-\delta)\cdot 1 + \delta\Pi^U(p_{1,t-1})\right).
\end{aligned}$$

Rearranging,

$$\begin{aligned}
V_2(p_{1,t}, p_{2,t}) &= \frac{1 - p_{1,t}}{1 - p_{1,t-1}}\left((1-\delta) + \delta\Pi^U(p_{1,t-1})\right), \\
\frac{p_{1,t} - p_{1,t-1}}{1 - p_{1,t}}\delta\Pi^I(p_{2,t-1}) &= (1-\delta)(1 - f(p_{1,t-1})) + \delta\left(\Pi^U(p_{1,t-1}) - V_2(p_{1,t-1}, p_{2,t-1})\right).
\end{aligned}$$

We now proceed as in section 5.2, defining $q_i = 1/(1-p_i)$, $w_i = V_i(p_1,p_2)/(1-p_{-i})$, and [slightly abusing notation] $f(q_i) = 1-\delta+\delta\Pi^U(1-1/q_i)$. Then we can rewrite the dynamics above as a set of four difference equations

$$\begin{cases} q_{1,t+1}-q_{1,t} = \frac{q_{1,t}f(q_{1,t})-w_{2,t}}{\Pi^I(q_{2,t})} \\ q_{2,t+1}-q_{2,t} = \frac{q_{2,t}\left(f(q_{2,t})-(1-\delta)(1-f(q_{1,t})(1+q_{1,t}^{-1}-q_{2,t}^{-1}))\right)-\delta w_{1,t}}{\delta\Pi^I(q_{1,t})} \\ w_{1,t+1} = q_{2,t}\left(f(q_{2,t})+(1-\delta)(q_{1,t}^{-1}-q_{2,t}^{-1})\right) \\ w_{2,t+1} = q_{1,t}f(q_{1,t}). \end{cases}$$

Observe that if q_1 and q_2 are on the boundary defined above, and if w_1 and w_2 are the corresponding pooling values, then we get a fixed point for the entire system (i.e. the RHS of the first two equations reduces to zero). We can again write this as a pair of second-order difference equations, if we prefer, from which monotonicity is more readily apparent. Computing the Jacobian evaluated at the critical point, we get:

$$\begin{bmatrix} 1+\frac{f(q_1)+q_1f'(q_1)}{\Pi^I(q_2)} & 0 & 0 & -\frac{1}{\Pi^I(q_2)} \\ q_2(1-\delta)\frac{f'(q_1)(1+q_1^{-1}-q_2^{-1})+(1-f(q_1))/q_1^2}{\delta\Pi^I(q_1)} & D & -\frac{1}{\Pi^I(q_1)} & 0 \\ -(1-\delta)q_2/q_1^2 & f(q_2)+(1-\delta)q_1^{-1}+q_2f'(q_2) & 0 & 0 \\ f(q_1)+q_1f'(q_1) & 0 & 0 & 0 \end{bmatrix}$$

where

$$D = 1+\frac{f(q_2)-(1-\delta)(1-f(q_1)(1+q_1^{-1}-q_2^{-1}))+q_2\left(f'(q_2)-(1-\delta)(1-f(q_1)/q_2^2\right)}{\delta\Pi^I(q_1)}.$$

We can relabel the Jacobian as

$$\begin{bmatrix} A & 0 & 0 & -B \\ C & D & -E & 0 \\ -F & G & 0 & 0 \\ H & 0 & 0 & 0 \end{bmatrix}$$

which has eigenvalues $\left\{\frac{1}{2}(A\pm\sqrt{A^2-4BH}, \frac{1}{2}(D\pm\sqrt{D^2-4EG}\right\}$. Now $A^2-4BH = (1+2x+x^2)-4x = (1-x)^2$, where $x = BH = \frac{f(q_1)+q_1f'(q_1)}{\Pi^I(q_2)}$, so the first two eigenvalues simplify to 1 and BH. The product of the remaining conjugate roots is $\frac{1}{4}(D^2-(D^2-4EG)) = EG$, but since $f(q_i) = 1-\delta+\delta\Pi^U(1-1/q_i) > 1-\delta > \Pi^I(q_{-i})$ and $f'(q_i) > 0$, both $BH > 1$ and $EG > 1$. Thus at least one of the conjugate pair must have norm greater than 1, and overall we have at least two eigenvalues larger than 1, with one unit root. The inverse matrix (since we were working in reverse time) thus has two eigenvalues less than 1, so that we have two stable dimensions. There is a center manifold from the unit root, but all that we require is at least one stable dimension. Thus, in a neighborhood of the boundary, given q_1 *and* a point on the boundary, we can find q_2, w_1, w_2 such that the system will converge to that point. Reverting to our standard notations and working backward (we use monotonicity here, which is easy to check), this says that from any initial p_1 and p_2, we can find values V_1 and V_2 so that the system continuously converges to some point on the boundary (with the appropriate pooling values as the limit). This completes the proof of existence. ■